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Supersymmetric quantum mechanics based on higher excited states

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Abstract. The formalism and the techniques of the supersymmetric (SUSY) quantum mechanics is generalized to the cases where the superpotential is generated/defined by higher excited eigenstates. The generalization is technically almost straightforward but physically quite non-trivial since it yields an infinity of new classes of SUSY-partner potentials, whose spectra are exactly identical except for the lowest $(m+1)$ states, if the superpotential is defined in terms of the $(m+1)$ eigenfunction, with $m=0$ reserved for the ground state. It is shown that in case of the infinite one-dimensional (1D) potential well nothing new emerges (the partner potential is still of Pöschl–Teller type I, for all m), whilst in case of the 1D harmonic oscillator we get a new class of infinitely many partner potentials: for each m the partner potential is expressed as the sum of the quadratic harmonic potential plus rational function, defined as the derivative of the ratio of two consecutive Hermite polynomials. These partner potentials of course have m singularities exactly at the locations of the nodes of the generating $(m+1)$ wavefunction. The SUSY formalism applies everywhere between the singularities. A systematic application of the formalism to other potentials with known spectra would yield an infinitely rich class of 'solvable' potentials, in terms of their partner potentials. If the potentials are shape invariant they can be solved at least partially and new types of analytically obtainable spectra are expected.

1. Introduction

After the classical papers of Witten (1981) and Gendenshtein (1983) the methods of supersymmetric (SUSY) (non-relativistic) quantum mechanics have quickly developed and it has been realized that: (1) there exist partner potentials with precisely the same energy spectra except for the ground state ($m=0$) (whose wavefunction $\phi(x)=\psi_0(x)$ is used to generate/define the superpotential $W(x)$, see below)‡, and that (2) if they are '*shape invariant*' their spectra and wavefunctions can be exactly and analytically solved. It is believed that the list of such shape invariant partner potentials is now complete and finite (Lévai 1989, Barclay *et al* 1993), and therefore quite limited in use. The research has been later further developed also in the direction of applying the WKB methods to such classes of Hamiltonians, including the search for improved simple quantization conditions which would be exact in the case of SUSY shape-invariant potentials (Barclay *et al* 1993, Barclay and Maxwell 1991, Barclay 1993, Inomata *et al* 1993, Junker 1995, Robnik and Salasnich 1996), and also in the direction of exploring the applicability of the path integral techniques (Inomata and Junker 1993, 1994). One of the nicest presentations of SUSY quantum

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‡ The ground-state energy $E_0^{(-)}$ is missing in the partner Hamiltonian H_+, so that its groundstate $E_0^{(+)}=E_1^{(-)}$.

mechanics was published by Dutt, Khare and Sukhatme (1988), henceforth referred to as DKS. We will use their notation. It should be mentioned at this point that the ideas involved behind the SUSY property and shape invariance were formulated first by Infeld and Hull (1951), where they were called the 'factorization method', and these authors refer further to the related ideas in the works of Schrödinger (1940, 1941).

2. Generalized supersymmetric formalism

The main point of this short paper is to point out that the whole formalism of the SUSY quantum mechanics can be generalized to arbitrary higher excited eigenstates $\phi(x) = \psi_m(x)$, $m = 0, 1, 2, \ldots$, used to generate the superpotential $W(x)$, namely

$$W(x) = -\frac{\hbar}{\sqrt{2\mu}} \frac{\phi'}{\phi} \tag{1}$$

where $\phi'(x) = \mathrm{d}\phi/\mathrm{d}x$, μ is the mass of the particle moving in the V_- potential, $2\pi\hbar$ is the Planck constant and m is the quantum number equal to the number of nodes of the eigenfunctions $\psi_m(x)$ of the starting potential $V_-(x)$. The energy scale is adjusted so that the $(m+1)$ energy eigenvalue is exactly zero, $E_m^{(-)} = 0$. The corresponding Hamiltonian is

$$H_- = -\frac{\hbar^2}{2\mu} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x)$$

and the Schrödinger equation reads

$$H_- \psi_m^{(-)} = H_- \phi = \left(-\frac{\hbar^2}{2\mu} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x) \right) \phi = 0. \tag{2}$$

Obviously, because $\phi'(x) \neq 0$ at the nodes y_j, the superpotential $W(x)$ will have singularities at the nodes y_j, $j = 1, 2, \ldots, m$, of ϕ. However, this does not invalidate our derivation, but it merely means, as will become clear later, that the partner potential generated by ϕ diverges to $+\infty$ when $x \to y_j$, for any $j = 1, 2, \ldots, m$. This implies that the potential wells are well defined between two consecutive singularities and that they do not communicate with solutions in the neighbouring wells. Thus, if $m = 0$, we have the common case of usual SUSY potentials defined on $(-\infty, +\infty)$, if $m = 1$ we have two separated potential wells, each of them on a semiinfinite domain, for $m = 2$ we have one infinite potential well on a finite domain between two nodes y_1 and y_2, and two binding potential wells on the two seminfinite domains $(-\infty, y_1]$ and $[y_2, +\infty)$, and so on. The (partner) potentials constructed in this way are non-trivial and certainly very interesting since they contribute to our list of solvable potentials which now becomes truly very rich and infinite in its contents.

In order to make this paper self-contained I will build up the formalism necessary to construct the partner potentials and to define the shape invariance, following DKS, in order to demonstrate that the SUSY formalism does not break down anywhere on its domain of definition, and to define the language needed to talk about further results that I shall present in this contribution.

First, we express the starting potential $V_-(x)$ in terms of the $(m+1)$st eigenfunction $\phi(x) = \psi_m(x)$, by solving (2)

$$V_-(x) = \frac{\hbar^2}{2\mu} \frac{\phi''}{\phi} \tag{3}$$

which is regular everywhere, because at the nodes y_j the second derivative $\phi''(x) = \mathrm{d}^2\phi/\mathrm{d}x^2$ also vanishes with ϕ. Thus, the basic Hamiltonian H_- reads

$$H_- = \frac{\hbar^2}{2\mu}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\phi''}{\phi}\right). \tag{4}$$

The two important operators are

$$A^+ = \frac{\hbar}{\sqrt{2\mu}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\phi'}{\phi}\right) \tag{5}$$

and

$$A = \frac{\hbar}{\sqrt{2\mu}}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\phi'}{\phi}\right) \tag{6}$$

which gives

$$H_- = A^+ A. \tag{7}$$

We further define the partner Hamiltonian H_+ and the partner potential V_+ as

$$H_+ = AA^+ = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x) \tag{8}$$

where

$$V_+(x) = V_-(x) - \frac{\hbar^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi'}{\phi}\right) \tag{9}$$

or

$$V_+(x) = -V_-(x) + \frac{\hbar^2}{\mu}\left(\frac{\phi'}{\phi}\right)^2. \tag{10}$$

The potentials V_+ and V_- are called *SUSY-m partner potentials*. We will show that they have the same energy levels, except for the $(m+1)$ lowest states of V_- for which there are no corresponding states of V_+, so that the ground state of the latter is $E_0^{(+)} = E_{m+1}^{(-)}$. All higher states then have identical energies. From equation (10) we see explicitly that at every node y_j, $j = 1, 2, \ldots, m$, of the defining eigenstate $\phi = \psi_m^{(-)}$ the partner potential(s) will have a singularity of the type $1/(x - y_j)^2$ such that $V_+(x) \to +\infty$ when $x \to y_j$, so that every branch of the partner potential will be confining up to infinity, and the solutions in various branches do not communicate. Thus, for each m we shall find $(m+1)$ (branches of the) partner potentials.

In terms of the superpotential W defined in equation (1) we can write

$$\phi(x) = \psi_m^{(-)}(x) = \exp\left(-\frac{\sqrt{2\mu}}{\hbar}\int^x W(x)\,\mathrm{d}x\right) \tag{11}$$

which is well defined in the definition domain of any of the branches of the partner potential, and obviously ϕ will go to zero where W has the poles $1/(x - y_j)$, as it should happen.

Some of the relationships can now be rewritten/reformulated in terms of the superpotential $W(x)$

$$\begin{aligned} A^+ &= -\frac{\hbar}{\sqrt{2\mu}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \\ A &= \frac{\hbar}{\sqrt{2\mu}}\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \end{aligned} \tag{12}$$

and also, the commutator of the operators A and A^+ is

$$V_{\pm}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2\mu}} W'(x) \qquad W'(x) = \frac{\mathrm{d}W}{\mathrm{d}x}. \tag{13}$$

Furthermore, we observe

$$V_+ = V_- + \frac{2\hbar}{\sqrt{2\mu}} \frac{\mathrm{d}W}{\mathrm{d}x} \tag{14}$$

and also

$$[A, A^+] = \frac{2\hbar}{\sqrt{2\mu}} \frac{\mathrm{d}W}{\mathrm{d}x}. \tag{15}$$

Now we have all the tools at hand to show that the SUSY partner potentials V_- and V_+ are isospectral except for the lowest $(m+1)$ states of V_- which have no counterpart in V_+, so that its ground state is $E_0^{(+)} = E_{m+1}^{(-)}$.

The demonstration, following DKS, is very easy. First we find that if $\psi_n^{(-)}$ is an eigenfunction of H_- with the eigenenergy $E_n^{(-)}$, then $A\psi_n^{(-)}$ is an eigenfunction of H_+ with the same energy:

$$H_+(A\psi_n^{(-)}) = AA^+A\psi_n^{(-)} = AH_-\psi_n^{(-)} = AE_n^{(-)}\psi_n^{(-)} = E_n^{(-)}A\psi_n^{(-)}. \tag{16}$$

Now we show that this applies only to the eigenstates n higher than m, $n = m+1, m+2, \dots$, by considering the normalization condition, by writing the normalized state $\psi_n^{(+)} = C_n A\psi_n^{(-)}$, and calculating the normalizing coefficient C_n,

$$\|\psi_n^{(+)}\| = C_n^2\langle A\psi_n^{(-)}|A\psi_n^{(-)}\rangle = C_n^2\langle\psi_n^{(-)}|A^+A\psi_n^{(-)}\rangle = C_n^2 E_n^{(-)}\|\psi_n^{(-)}\|. \tag{17}$$

If all $\psi_n^{(-)}$ are normalized (they are certainly orthogonal, because we deal with one-dimensional systems, where degeneracies are forbidden due to the Sturm-Liouville theorem (Courant and Hilbert 1968) and, therefore, all eigenstates must be orthogonal), then

$$C_n = \frac{1}{\sqrt{E_n^{(-)}}} \tag{18}$$

which implies that the construction succeeds iff $E_n^{(-)} > 0$, implying that $n > m$. Thus, the two Hamiltonians H_- and H_+ defined in (4) and in (8) are isospectral except for the lowest $(m+1)$ eigenstates of H_- which have no counterpart in H_+.

Now counting the eigenstates of H_+ from $n = 0, 1, 2, \dots$, where $n = 0$ is the ground state and n is the number of nodes of the (now also normalized) eigenfunction $\psi_n^{(+)}$, we have

$$\psi_n^{(+)} = \frac{1}{\sqrt{E_{m+1+n}^{(-)}}} A\psi_{m+1+n}^{(-)} \qquad E_n^{(+)} = E_{m+1+n}^{(-)}. \tag{19}$$

Of course it is easy to show that, conversely, for every eigenstate $\psi_n^{(+)}$ of H_+ there exists the normalized eigenstate of H_-, namely

$$\psi_{m+1+n}^{(-)} = \frac{1}{\sqrt{E_n^{(+)}}} A^+\psi_n^{(+)} \qquad n = 0, 1, 2, \dots. \tag{20}$$

This completes our proof of isospectrality, generalized to the case that the generating function ϕ of the superpotential W, defined in equation (1), is a higher excited wavefunction, namely $\phi = \psi_m^{(-)}$, $m = 0, 1, 2, \dots$. As we have seen, the formalism of superpotential and of the partner potentials works everywhere except at the singularities located at the nodal points y_i of ϕ, where the partner potential V_+ goes to infinity as $1/(x-y_i)^2$, thereby defining several branches of V_+ well defined on their disjoint domains of definition.

We have demonstrated that if one of the partner systems (the Hamiltonians) can be solved completely (by calculating the energy levels and the eigenfunctions), then the SUSY formalism enables one to solve the partner problem completely, following equation (19). One of the most important cases is, of course, the harmonic oscillator, which we will discuss in detail below.

If the solutions for the two partner Hamiltonians are both unknown, then another approach is necessary to solve them. In the case of the standard SUSY formalism with $m = 0$ we have the important class of the shape invariant potentials. As is well known (DKS) the shape invariance of the two partner potentials V_- and V_+ is defined by

$$V_+(x; a_0) = V_-(x; a_1) + R(a_1) \tag{21}$$

where a_0 is a set of parameters, $a_1 = f(a_0)$ and $R(a_1)$ is independent of x. The procedure is now (essentially embodied in the factorization method of Infeld and Hull (1951)) the following. Consider a series of Hamiltonians $H^{(s)}$, $s = 0, 1, 2, \ldots,$ where $H^{(0)} = H_-$ and $H^{(1)} = H_+$, by definition

$$H^{(s)} = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x; a_s) + \sum_{k=1}^{s} R(a_k) \tag{22}$$

where

$$a_s = f^s(a_0) = \underbrace{f \circ \cdots \circ f}_{s}(a_0). \tag{23}$$

Now compare the spectra of $H^{(s)}$ with $H^{(s+1)}$, and find

$$\begin{aligned} H^{(s+1)} &= -\frac{\hbar^2}{2\mu} + V_-(x; a_{s+1}) + \sum_{k=1}^{s+1} R(a_k) \\ H^{(s+1)} &= -\frac{\hbar^2}{2\mu} + V_+(x; a_s) + \sum_{k=1}^{s} R(a_k). \end{aligned} \tag{24}$$

Thus it is obvious that $H^{(s)}$ and $H^{(s+1)}$ are SUSY partner Hamiltonians, and they have the same spectra from the first level upwards except for the ground state of $H^{(s)}$ whose energy is

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k). \tag{25}$$

When going back from s to $(s-1)$ we reach $H^{(1)} = H_+$ and $H^{(0)} = H_-$, whose ground-state energy is zero and its nth energy level is coincident with the ground state of the Hamiltonian $H^{(n)}$, $n = 1, 2, \ldots$. Therefore, the complete spectrum of H_- is

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \qquad E_0^{(-)} = 0. \tag{26}$$

The generalization of shape invariance to the case of any $m \geqslant 0$ is straightforward, but it results in higher complexity and, therefore, it is more rarely satisfied by the specific systems. By repeating the above arguments we reach the conclusion that, when (21) is satisfied for a superpotential W with given m, then we cannot calculate the entire spectrum of the shape invariant potential/Hamiltonian H_-, but only the subset (subsequence) of period $m+1$, namely

$$E^{(-)}_{m+n(m+1)} = \sum_{k=1}^{n} R(a_k) \qquad E^{(-)}_m = 0, n = 1, 2, \ldots. \tag{27}$$

In the special case $m = 0$ we, of course, recover the formula (26). For $m > 0$ we have *no example* of SUSY-m shape invariance so far.

3. The example of the harmonic oscillator

Let us consider a few examples of SUSY-m partner potentials. First, consider the harmonic oscillator, defined by

$$V_-(x) = \tfrac{1}{2}\mu\omega^2 x^2 - (m + \tfrac{1}{2})\hbar\omega \tag{28}$$

shifted in energy so that

$$E^{(-)}_m = 0. \tag{29}$$

Introducing the natural unit of length α we can write the ground-state wavefunction as

$$\psi_0(x) = \frac{1}{\pi^{1/4}\alpha^{1/2}} \exp\left(-\frac{x^2}{2\alpha^2}\right) \qquad \alpha = \sqrt{\frac{\hbar}{\mu\omega}}. \tag{30}$$

Defining the creation operator a^+,

$$a^+ = \frac{1}{\sqrt{2}}\left(-\alpha\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{\alpha}\right) \tag{31}$$

we can write down all the eigenfunctions, in particular the $(m+1)$ one, labelled by m and denoted by ϕ, as follows

$$\psi^-_m(x) = \frac{(a^+)^m}{\sqrt{m!}}\psi_0(x) = \phi(x). \tag{32}$$

Now we calculate the superpotential according to equation (1), by using the following operator when calculating $\phi' = \mathrm{d}\phi/\mathrm{d}x$, obtained from equation (31),

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\alpha}\left(\frac{x}{\alpha} - \sqrt{2}a^+\right) \tag{33}$$

and find

$$W(x) = -\frac{\hbar}{\alpha\sqrt{2\mu}}\left(\frac{x}{\alpha} - \sqrt{2(m+1)}\,\frac{\psi_{m+1}(x)}{\psi_m(x)}\right). \tag{34}$$

Using the explicit solution for $\psi_m(x)$, namely

$$\psi_m(x) = \left(\frac{1}{2^m m!}\sqrt{\frac{1}{\pi\alpha^2}}\right)^{1/2} \boldsymbol{H}_m\left(\frac{x}{\alpha}\right) \exp\left(-\frac{x^2}{2\alpha^2}\right) \tag{35}$$

where $\boldsymbol{H}_m(z)$ is the Hermite polynomial (Abramowitz and Stegun 1965), we obtain

$$\frac{\psi_{m+1}(x)}{\psi_m(x)} = \frac{1}{\sqrt{2(m+1)}}\,\frac{\boldsymbol{H}_{m+1}(x/\alpha)}{\boldsymbol{H}_m(x/\alpha)} \tag{36}$$

and hence

$$W(x) = -\frac{\hbar}{\alpha\sqrt{2\mu}}\left(\frac{x}{\alpha} - \frac{\boldsymbol{H}_{m+1}(x/\alpha)}{\boldsymbol{H}_m(x/\alpha)}\right). \tag{37}$$

From this and equation (14) we get finally the SUSY-m partner potential of the harmonic potential (28), namely

$$V_+(x) = \frac{1}{2}\mu\omega^2 x^2 - \left(m + \frac{3}{2}\right)\hbar\omega + \hbar\omega \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\boldsymbol{H}_{m+1}(z)}{\boldsymbol{H}_m(z)}\right)_{z=x/\alpha} \tag{38}$$

whose energy levels are the same as for the (energy shifted) harmonic oscillator, namely

$$E_n^{(+)} = (n+1)\hbar\omega \qquad n = 0, 1, 2, \ldots. \tag{39}$$

so that the ground-state energy $E_0^{(+)} = \hbar\omega$, which is equal to the $(m+1)$ energy level of (28).

This result is important, because for $m > 0$ it yields new interesting potentials with purely discrete spectrum, isospectral to the harmonic oscillator, except for the lowest $(m+1)$ eigenstates. Let us look just at the few lowest cases.

$$\begin{aligned}
m = 0 \qquad & V_+(x) = \frac{1}{2}\mu\omega^2 x^2 + \frac{1}{2}\hbar\omega \\
m = 1 \qquad & V_+(x) = \frac{1}{2}\mu\omega^2 x^2 - \frac{1}{2}\hbar\omega + \hbar\omega\frac{\alpha^2}{x^2} \\
m = 2 \qquad & V_+(x) = \frac{1}{2}\mu\omega^2 x^2 - \frac{3}{2}\hbar\omega + 4\hbar\omega\left(\frac{1}{2z^2-1} + \frac{2}{(2z^2-1)^2}\right) \qquad z = \frac{x}{\alpha} \\
m = 3 \qquad & V_+(x) = \frac{1}{2}\mu\omega^2 x^2 - \frac{5}{2}\hbar\omega + \frac{3\hbar\omega}{z^2}\left(1 + \frac{6}{2z^2-3} + \frac{12}{(2z^2-3)^2}\right) \qquad z = \frac{x}{\alpha}.
\end{aligned} \tag{40}$$

The case $m = 0$ is the usual SUSY-0 case showing just that the 1D harmonic oscillator potential is indeed SUSY-0 shape invariant. However, $m = 1$ gives a new example, which nevertheless is well known as the radial problem of the three-dimensional harmonic potential, which is thus the SUSY-1 partner potential of the 1D harmonic oscillator potential. It has only one singularity at $x = 0$. Next, for $m = 2$ in the above list, we see the first new non-trivial example, of a specific rational potential which is the SUSY-2 partner potential of the 1D harmonic potential. It has singularities at the two nodes $x = y_1 = -\alpha/\sqrt{2}$ and $x = y_2 = +\alpha/\sqrt{2}$ of the type $1/(x - y_j)^2$. Therefore it has three branches (ranges), namely $(-\infty, y_1]$, $[y_1, y_2]$ and $[y_2, +\infty)$. The spectrum is identical in each of them. Furthermore, in case $m = 3$, we have three singularities at the nodes where x is equal to $y_1 = -\alpha\sqrt{3/2}$, $y_2 = 0$ and $y_3 = -y_1 = +\alpha\sqrt{3/2}$, and thus we have two independent different potentials within the two ranges $[y_2, y_3]$ and $[y_3, +\infty)$. (The other two ranges confine the potentials which are equivalent due to the evenness of $V_+(x)$.)

For higher m we obtain new classes of interesting rational potentials, all of them being isospectral to the harmonic oscillator except for the lowest $(m+1)$ eigenstates of the latter, which are missing in the partner potentials. For each m we have rational potentials with $(m+1)$ branches, defined by the m nodes y_j, $j = 1, \ldots, m$. Since the Hermite polynomials $\mathrm{H}_m(z)$ are even or odd functions of z, depending on whether m is even or odd, the superpotential (37) is always odd function of x and, therefore, the SUSY-m partner potential $V_+(x)$ in equation (38) is always an even function of x with m singularities.

Asymptotically when $|x| \to \infty$ the potential still behaves as an harmonic quadratic potential with the leading term $\frac{1}{2}\mu\omega^2 x^2$, which is true for any m, as can be shown using the asymptotic properties of the Hermite polynomials, as described below.

The limiting (semiclassical) behaviour of the potential V_+ when $\hbar \to 0$ is interesting. It implies that $z = x/\alpha = x\sqrt{\mu\omega/\hbar}$ tends to $+\infty$ and, therefore, from the asymptotic properties of the Hermite polynomials $\mathrm{H}_{m+1}(z)/\mathrm{H}_m(z) \to 2z$, for $z \to +\infty$, we conclude

$$V_+(x) \longrightarrow \tfrac{1}{2}\mu\omega^2 x^2 - (m - \tfrac{1}{2})\hbar\omega \longrightarrow \tfrac{1}{2}\mu\omega^2 x^2 \qquad \text{when } \hbar \to 0. \tag{41}$$

Thus the semiclassical limiting form of all these potentials is just the harmonic quadratic potential, meaning that all the rational potentials in (38) all have zero classical limit. From (38) it is clear that the harmonic oscillator potential is *not* SUSY-m shape invariant, except for $m = 0$, which is the familiar case of shape invariance (see DKS).

4. Discussion and conclusions

Using the same formalism applied to known solvable potentials for various m we can systematically construct the vast class of new potentials which will be isospectral to each of the known solvable potentials, almost all of them being SUSY-0 shape invariant, and listed in DKS.

Finally, we can state the result which can be easily verified in a straightforward manner (we omit the derivation due to the lack of space here), that the Pöschl–Teller type I potential is the SUSY-m partner potential of the infinite potential well *for any value of* $m \geqslant 0$. At present, we do not know any specific cases of SUSY-m shape invariance with $m > 0$, and also have no further calculations for SUSY-m partner and solvable potentials, which remains as a future project.

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